Trilateration and the GPS

Students will be introduced to the science behind the GPS. They will be exposed to the principles of trilateration and its application to the most modern way of finding your location. Students will simulate the principles of the GPS using an interactive map in order to locate their position using a number of fictional satellites.

Emmanuel Chaniotakis, Physicist
Research and Development Department, Ellinogermaniki Agogi

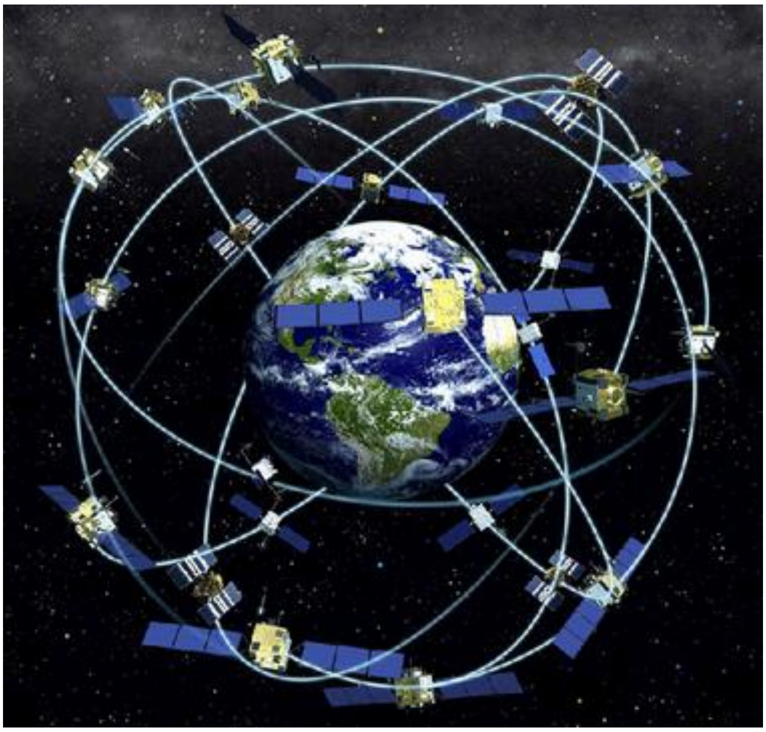
<u>Orienting & Asking Questions</u>

<u>Orienting: Provide Contact with the content and/or provoke curiosity (30 mins)</u>

Nowadays, everyone who wishes to find their location on a map or ask for directions concerning the navigation towards a specific destination, use the : Global Positioning System (or in short: GPS).

Originally constructed in the 70's by the USA Department of Defence for military purposes, the GPS can be found everywhere: from cell phones, to bulldozers, wristwatches and other gadgets.
Every technological unit which needs precise timing and location information uses GPS.
Scientific instruments, military equipment, communication networks, transportation, financial markets and power grids are only a few examples of applications depending heavily on GPS for precise time synchronization.

The first question that comes into mind is: "How does the GPS work? And how does it achieve such a tremendous impact on our lives?"
Using a constellation of GPS satellites (31 of them up to August 17, 2015), every location on earth is simultaneously monitored by 5 satellites so that one can obtain a stigma of their location with an accuracy that can reach 3 to 10 m. Considering the fact that the satellites are orbiting at a 20km distance from the earth's surface you can imagine the precision of these instruments!



Watch the following video (up to 3:07) to get introduced to the functionality of the GPS:

https://www.youtube.com/watch?v=_vfzAL5L29Y

For the teacher

You can visit the following link for an overview of the GPS functionality:

http://www.montana.edu/gps/understd.html#Calculating_a_Position

As an alternative to the video provided, you can show the following video to your students in order to introduce them to the GPS:

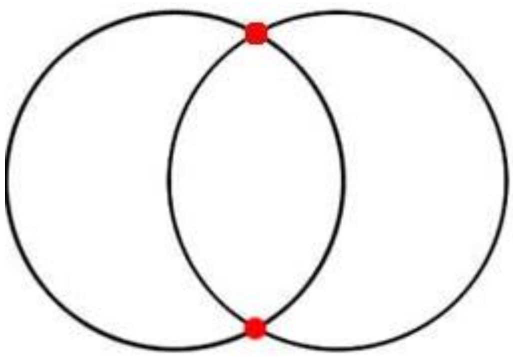
https://www.youtube.com/watch?v=uotknd6hlxk

Define Goals and/or questions from current knowledge

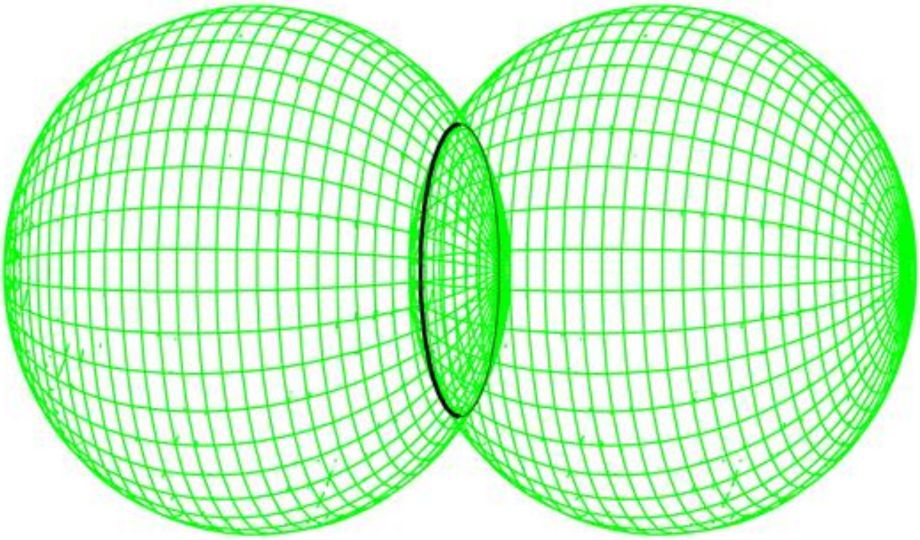
In order to comprehend in depth the principles governing the function of the GPS, we need to review some fundamental geometry and physics concepts:

Intersection of two circles

Two circles of different radii can intersect having thus two points in common:



If we go to 3 dimensions, two spheres intersect in the following way:



Can you define the common territory between the two spheres? Is it a point or something else? What would happen if we had a third sphere intersecting with the two?
How many common points would there be?

For the teacher:
The common territory between two intersecting spheres is a circle. If we add a third sphere, then the three of them have two points in common. This property is of great importance when we apply the trilateration technique to measure distances with the GPS.

Uniform Linear Motion

Let's start by remembering the all time classic relationship for uniform linear motion –i.e the motion of an object with constant velocity.
Time, distance and velocity are related by the following equation :

$$x = u \cdot t$$

In order to find the value of one of the three variables, the other two must be known from before.

$$c = 299792458\ m/s$$

Can you think of some everyday examples in which we observe uniform motion? Discuss them with your classmates.

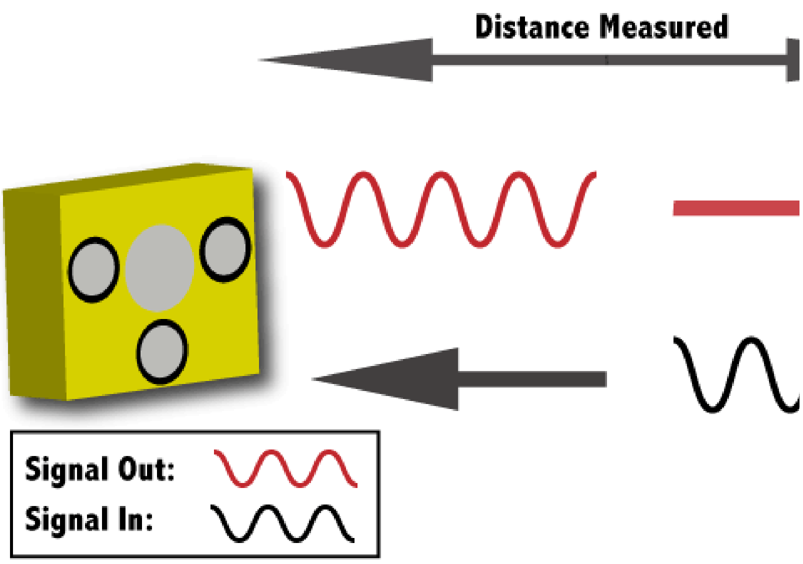
The speed of Light

Light travels with constant speed, which differs from the one medium of propagation to the other. In vacuum, the speed of light is:

In other mediums, the speed of light equals to the value stated above, divided by a number which is larger than 1 and differs from medium to medium (it is called the index of refraction and depends on the properties of each medium).

The speed of light in vacuum is the highest speed in the Universe, and is a constant. This constancy is very important for "Time of Flight" measurements. When we want to measure the distance between two points, a simple way to do so is to place a light source in the first point and a detector in the other point. If we measure the time needed for the light to go from A to B, then we can multiply by the speed of light in order to obtain the distance between the two points.

$$t = \frac{2X}{c}$$



The above picture shows a more realistic implementation of the time of flight principle. A light source (yellow box) emits a light pulse which reflects on the black object. A clock starts counting time from the moment of the emission and on. The light reflects and goes back to the yellow box where it is detected. Its arrival time is recorded.

If we define the unknown distance between the two objects as X , the total time elapsed between emitting and receiving the signal at this example is:

Thus, after measuring time and replacing the known value of the speed of light, we can obtain the value of the distance. (The factor two appears because light travels the distance twice, once to and once from the black object).

Can you think of any everyday application of the time of flight technique?
What do you think? Is it possible to perform the same technique without using light but another probe instead? Discuss your views.

For the teacher:

You can show this video to the students concerning time of flight:
https://www.youtube.com/watch?v=r2VZPl4j0X4

For further information concerning time of flight you can check the following :
https://en.wikipedia.org/wiki/Time_of_flight

This is the way time of flight is used by police to identify the speed of a vehicle:
https://en.wikipedia.org/wiki/LIDAR_speed_gun

What do you think concerning the relevance of the above knowledge to the GPS principle of operation? Proceed to the next step to find out!

Hypothesis Generation and Design (20mins)

Generation of Hypotheses or Preliminary Explanations

The GPS works using the method of trilateration.

Let's see what this is all about:

Trilateration is the method used to find the unknown location of an object when its distance from at least three objects is known.

Trilateration is fully employed in the GPS and has numerous applications such as finding the epicenter of an earthquake.

How is it done? This is the centerpiece of our investigation!

For the teacher:

Trilateration is a general geometrical method which has numerous applications.

Once the students have gotten to comprehend the concept of trilateration using this educational scenario, they can easily jump to activities such as locating the epicenter of an earthquake and others.

For further information concerning trilateration and the GPS you can visit the following links:

https://www.youtube.com/watch?v=4O3ZVHVFhes

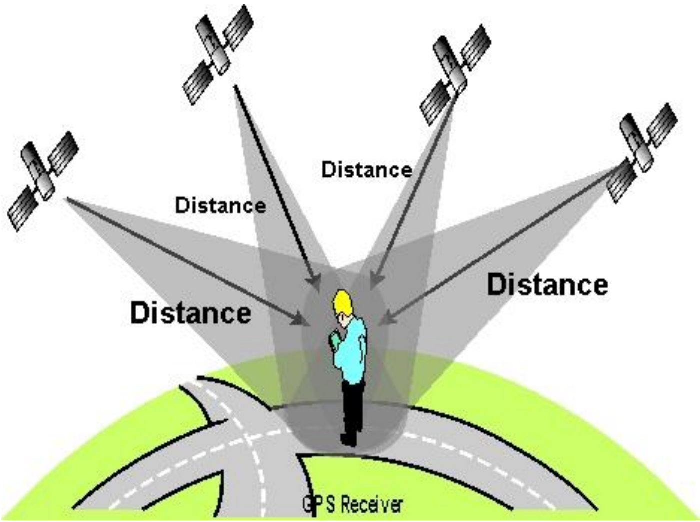
https://www.youtube.com/watch?v=PLjld-edVj8

Make sure that the students do not confuse trilateration with triangulation.

Trilateration is defined as the method employed in order to find your unknown location when you know its distance from at least three reference points.

Triangulation is the method employed in order to calculate the unknown distance between a reference baseline and the desired location, with the knowledge of the baseline length and the angles between the location and the two ends of the baseline.

https://www.youtube.com/watch?v=Nv_oiLPJOV0



As we have seen, the GPS employs a constellation of 31 satellites orbiting the earth at a distance of about 20km from the surface of the earth.

Each point of the earth is viewed by at least 4 satellites as we can see at the illustration above. Each satellite emits a signal which reaches our receiver, travelling at the speed of light .

In order to find the location of our GPS receiver, we need the signals of at least four satellites to intersect. The point of intersection is our location. This is the method of trilateration.

For the teacher:

We have intentionally made some simplifications:

For example the fact that the GPS receiver clock is in sync with the satellite clocks. In fact, the satellite contains atomic clocks which are far more precise and expensive than the clocks inside the simple commercial receivers.

If the students are not familiar with wave motion or electromagnetic waves, it is suggested that you don't emphasize this point. The students may also envision the radiowave emission as rays emitted by the satellite and expanding radially outwards.

Design/Model

In order to simulate the function of the GPS we will use a 2 dimensional model. This model will help us understand the principles of trilateration. The conclusions drawn from this activity can be directly applied in 3 dimensions and the real conditions of GPS operation.
We will be using a flat interactive map in which we shall mark the locations of the "GPS satellites". Using provided data we will find the distance from our location to the satellites and then we will perform trilateration in order to find where we are.

Can you explain what are the major differences between our 2 dimensional and a 3 dimensional model?

For the teacher:
Before you start the investigation part, make sure that the students are familiar with the terms latitude and longitude. If not, take a few minutes to remind them.
https://en.wikipedia.org/wiki/Latitude

https://en.wikipedia.org/wiki/Longitude

- **Planning and Investigation (45 mins)**

  Plan Investigation

  Divide in 3 groups. Each of the groups will be given the same coordinates of the GPS satellites' positions and different data that will be needed in order to calculate the distances between the satellites and your location.

  As a first step of your investigation, browse the interactive map here: https://mapmaker.nationalgeographic.org/

Get used to the controllers and the options of the map.

Click on the Latitude-Longitude options and choose to Show Longitude and Latitude position as demonstrated below:

Using the “Draw a marker” option you will flag the GPS satellites on the map:

Each satellite should be flagged with a marker of different number.

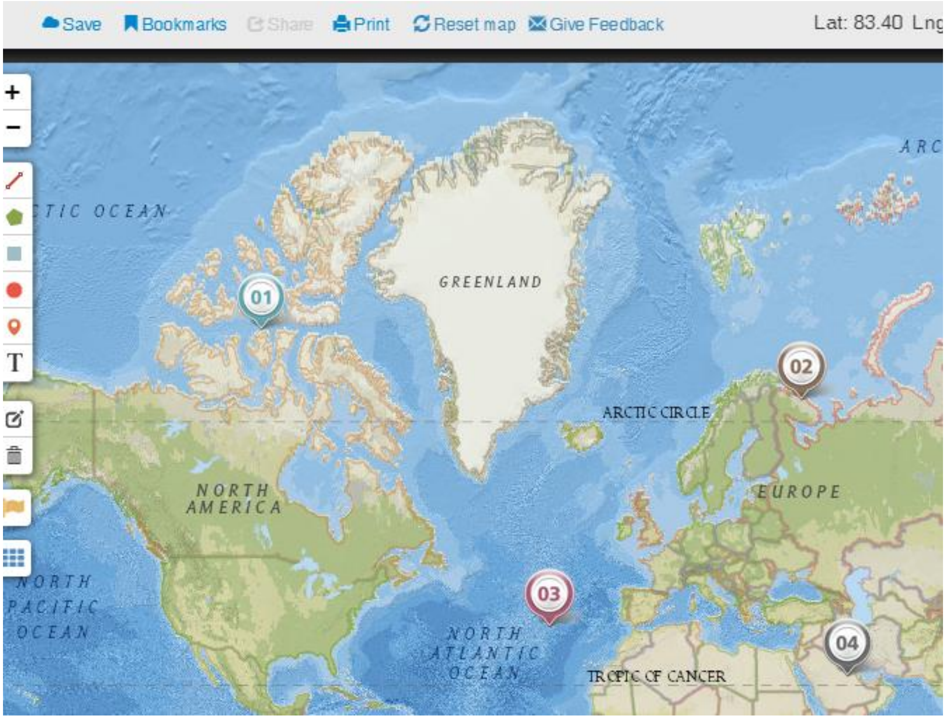
Make sure that you pinpoint the satellites at the exact spots provided.

Latitude and longitude will be provided to improve accuracy. If you make a mistake, then choose the option “Reset Map” and start all over. Use the zoom in and zoom out option for extra help.

The locations of the satellites are given at the following table:

Satellite Nr	Latitude (◦)	Longitude (◦)
1	41,82	4,31
2	54,53	1,65
3	49,72	-18,5

After you have flagged the GPS satellites, your map will look like this:



Perform Investigation

Now each team shall use the following data concerning the time elapsed from each satellite.

Team 1:

Satellite	Time Interval (sec)
1	0,003373901
2	0,001908954
3	0,006092715

Team 2:

Satellite	Time Interval (sec)
1	0,00605192
2	0,001786836
3	0,003399051

Satellite	Time Interval (sec)
1	0,006775321
2	0,004257879
3	0,009741639

- The first task of each team is to convert the time interval data in meters, using the known value of the speed of light given earlier :

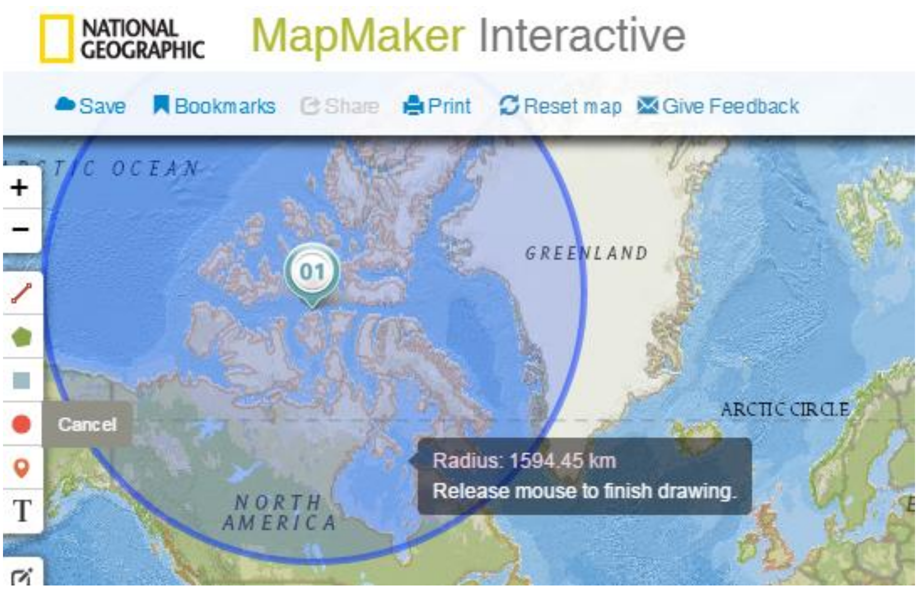
$$c = 299792458 \; m/s$$

Due to the precision of the numbers, a scientific calculator will be needed (or Microsoft Excell). Write your results at your notebook.

- Now, go back to the map and choose the “Draw a Circle “ option:

Starting from the position of each flag, expand the circle outwards until the distance displayed on your monitor equals the one you calculated already.

<u>Be extra careful because when you stop expanding the circle, because once you lift the cursor, the distance isn't displayed anymore.</u>



- Do the same for each satellite.

- Find the point in which the three circles intersect and flag it with a marker of your liking (here we have chosen a flag).

- Team 1 will mark location A, team 2 will mark location B and team 3 will mark location C.

- Write down the coordinates of the location you found as well as the country it belongs in.

Your final result should look like this:

You have now implemented successfully the trilateration technique!

For the teacher:
It is advised that you take some time to get used to the controllers, and especially the expanding circle and the value of its radius. Students might have a problem in this and could need extra time to repeat their investigation. You may also choose to propose different locations for the satellites and time interval data received. If you consider that the activity takes too long, you may as well choose to work with two locations instead of three.

Analysis & Interpretation (35 mins)

Analysis and interpretation : Gather result from data

In this section we will push one step further and calculate distances using the results we obtained so far.

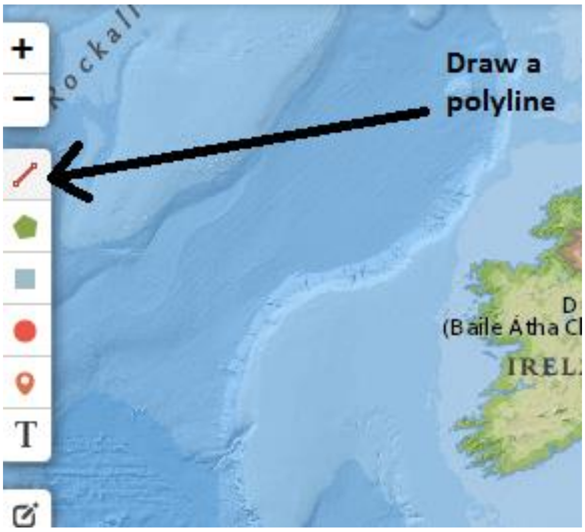
This is one of the major operations that can be done using the GPS and this way one can calculate the distance from a starting point to an ending point.

Every team has now obtained the latitude and longitude of their position. A representative of each team writes the latitude and longitude they measured on the blackboard.
All the teams use the interactive map and flag the three locations on it.

For the teacher:
If you consider that you have not enough time to carry out the measurement of all three distances, you may as well choose one and proceed with it.
The desired results are the following:

	Team 1				
Satellite Nr	**radius (km)**	**c (km/sec)**	**time interval (sec)**		
1	1011,47	299792,458	0,003373901	**Location A**	
2	572,29	299792,458	0,001908954	**Latitude(◦)**	**Longitude(◦)**
3	1826,55	299792,458	0,006092715	49,94	6,55
Satellite Nr	**Team 2**				
1	1814,32	299792,458	0,00605192	**Location B**	
2	535,68	299792,458	0,001786836	**Latitude(◦)**	**Longitude(◦)**
3	1019,01	299792,458	0,003399051	53,54	-6,57
Satellite Nr	**Team 3**				
1	2031,19	299792,458	0,006775321	**Location C**	
2	1276,48	299792,458	0,004257879	**Latitude(◦)**	**Longitude(◦)**
3	2920,47	299792,458	0,009741639	52,5	21,07

Now, you will use the option “Draw a polyline” of the map:



Your task now is to use the polyline in order to measure the distances between the flagged locations on the map.

You should be extra careful concerning the distance you measure. Save the number you obtain at the exact spot that you finished your line.

Now , use the distance calculator http://www.nhc.noaa.gov/gccalc.shtml in order to compute the actual distance between locations A, B and C. The calculator uses the same mathematical formula employed by the GPS in order to compute actual distances between locations.

For the teacher:

Make sure that the students use as starting and ending points the exact spots that they marked when they used the trilateration method. This can be done by zooming in and out of the map until the starting and ending coordinates correspond exactly to the desired points.

The results obtained using the two methods are summarized at the following table. As we can see, the accuracy of distance determination is of the order of 8km:

Distance	Polyline (km)	Distance Calculator (km)
AB	973,6	981
AC	1050,8	1043
BC	1828,76	1830

Compare the results obtained by the two methods and discuss the deviations you observe. Consider the distances calculated by the distance calculator as the optimal.

If you zoom in the intersection points of the three circles you drew to find your location, do they overlap at exactly one point?

Can you estimate the accuracy of the method you used in your investigation?

Compare the accuracy you estimated with the accuracy of a commercial GPS. What do you observe?

Name the experimental errors you consider most important and discuss ways to eliminate them.

For the teacher:

If you zoom in at the place where the three circles overlap, you will see that they do not coincide exactly at a point. A triangle is formed instead. If we measure the area

of the triangle and create a circle of the same area, its radius can be defined as the accuracy of our method (and is of the order of a few km depending on the zoom options we used. Higher resolution implies higher accuracy).
Therefore if we assume a 5km accuracy with the average distance between satellites and receiver being 1000km, we have an uncertainty of the order 0.5%.
In comparison, a real GPS satellite is 20km far from the surface of the earth with an error of up to 10m in determining the location of a receiver. Therefore in a real world example we have an uncertainty of the order 0.00005% which is four orders of magnitude better than the method we used!

Conclusion & Evaluation (30mins)

Conclude and communicate result/explanation

Discuss the difficulties you encountered during your investigation.
Did you understand the method of trilateration?

The details of the GPS functionality can be explained as follows:

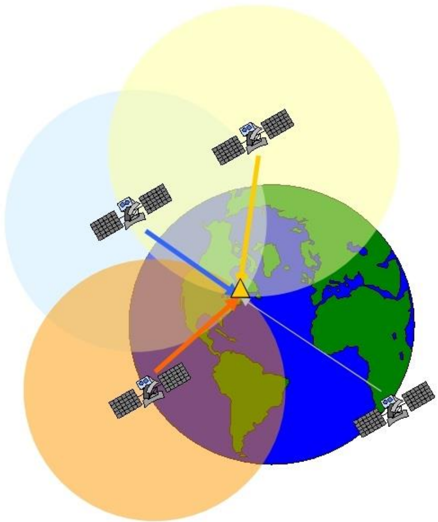
Each satellite emits radiowaves (which are actually light of lower frequency than the visible light) which travel at the speed of light.
These radiowaves carry information concerning the exact location of the satellite and the precise time (obtained using its internal atomic clock) it was when the radiowave was emitted. The clocks of all the satellites are synchronized.
You can imagine the radiowaves as concentric spheres expanding at the speed of light with the satellite in the middle. From now and on we will refer to the radiowaves from a satellite as "the signal".

Our well known GPS device contains a radio receiver which will receive the signal from each satellite it can view. It also contains an internal clock which is synchronized with the clocks of the satellites.

We consider that the radiowave has reached its destination when the spherical radiowave reaches the receiver.

The signal from the closest satellites will arrive earlier while the signal from the furthest ones will arrive later.

By subtracting the time the signal started from the satellite from the time the signal arrived at the receiver, we can obtain the time interval. This time interval can be converted to distance using the fact that the speed of light is a constant of known value.



This way we can obtain the distance between each satellite and the receiver.
The distance is equal to the radius of the sphere.

The point of intersection of the four spheres is our location. This is the method of trilateration.

Evaluation/Reflection

Discuss with your teacher and classmates the method of trilateration and its application to the GPS.
Exchange opinions concerning the impact of the GPS to our lives and the society.

For the teacher:
You can stimulate further discussion using information from the links below:

http://rachelgreenberg.tripod.com/id12.html
http://www.batteriesinaflash.com/blog/how-gps-technology-has-changed-society/

It would be optimal if the students could see a real time demonstration with a GPS if the teacher has access to one.

Take some time to answer the following quiz concerning the GPS :
http://electronics.howstuffworks.com/gadgets/travel/gps-quiz.htm

For the teacher:
The teacher is advised to take the quiz prior to the students so that the students can be guided properly. This is not an assessment quiz, rather a stimulant for a further discussion.

Consider other explanations

Think about how we could use the method of trilateration in other aspects of science and everyday life. Discuss it with your classmates.

Assessment using provided assessment questions

After students finished the last learning activity they answer a set of multiple choice assessment questions. These questions measure the problem solving skills of the students.

According to the 'Big Idea' that is focused in the Demonstrator it is possible to pick at least one matching Assessment Question out of the official catalogue of ISE assessment questions and to create up to three (minimum two) Assessment Questions by your own in order to investigate the problem solving competency of your students . It is possible to immediately compare the results of the students with the average PISA results.